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Optimal and competitive programs in reachable multi sector models*

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Summary. We show that in multi-sector optimal growth models, where the technology satisfies a simple reachability condition, infinite horizon programs which satisfy the competitive conditions are optimal. We provide examples of a variety of production models where the reachability condition is satisfied. An example is also provided where the reachability condition is not satisfied and there are competitive programs which are not optimal. The results of the paper are of interest from the standpoint of decentralization in intertemporal economies.

Keywords and Phrases: Competitive program, Optimality, Decentralization, Reachability.

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1 Introduction

We show that in multi-sector optimal growth models, where the technology satisfies a simple reachability condition, competitive programs are optimal. We elaborate on this below, beginning by placing the paper in the context of the existing literature and putting it in perspective.

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The relation between decentralized decision making and optimal allocation of resources, is a theme which is central to economics. It is the main thrust of the theorems of welfare economics, which establish a link between price systems and efficient allocation of resources in static or finite horizon planning problems (Koopmans, 1957). It is prominent in the literature on informationally decentralized mechanisms, following Hurwicz (1960). In capital theory, there is an extensive literature, pioneered by Malinvaud (1953), on price systems which characterize optimal allocations in intertemporal resource allocation problems with no terminal date.

Recent years have witnessed a revival of interest in decentralization in the context of intertemporal economies (see, for example, Hurwicz and Majumdar, 1988; Hurwicz and Weinberger, 1990; Majumdar, 1992). The central question addressed may be stated as follows. Suppose that a class of environments and evaluation criteria is given. Environment includes technology, tastes, and initial resources; and we shall suppose that programs are evaluated according to the discounted, or undiscounted, sum of utilities criterion. Is it possible to devise a set of behavior rules for individual agents (producers and consumers) with the following characteristics: that they involve separate decision making by producers and consumers; are myopic, involving decisions over finite horizon; are based upon private information and, perhaps, limited amounts of commonly available information (such as the history of evolution, prices); and have the characteristic that optimal allocations, and only the optimal ones, can be sustained by such decentralized behavior?[1]

It is well known, from earlier literature on optimal growth theory[2], that there is a close connection between price systems and optimal allocations which, in some respects, strongly resembles the relation between price systems and optimal allocations found in static or finite horizon problems; however, there is also a fundamental difference which stems from the infinite horizon nature of the problem. Optimal programs may be characterized by the existence of a sequence of present value prices $p(t)$, at which the time path of capital stocks $x(t)$ satisfies two distinct sets of requirements: (a) support properties for the technology and utility (see (2.1) and (2.2) in Section 2 for precise statements); and (b) some appropriate kind of limiting behaviour of the present value of capital $p(t)x(t)$.[3] The set of rules in (a), usually referred to as the "competitive conditions", are analogous to those in static optimality problems. These competitive conditions

[1] In the language of Hurwicz (1972), the question posed is whether there is an allocation mechanism which is unbiased and non-wasteful. Private information refers to information specific to individual agents: their own technology, utility function and the like.

[2] See, for instance, Gale (1967), Weitzman (1973) and McKenzie (1986), for treatments of the "Reduced Form Model", where consumption is implicit and utilities are defined on initial and terminal stocks of capital; and Peleg (1970),(1974), Peleg and Ryder (1972) and Peleg and Zilcha (1977) for treatments of a model where consumption appears explicitly. Malinvaud's (1953) paper deals with efficient consumption programs; he is not concerned with programs maximizing a discounted or undiscounted sum of utilities. For a unified treatment, see also Cass and Majumdar (1979).

[3] The relevant condition when utilities are discounted is "$p(t)x(t)$ converges to 0 as $t \to \infty$". If utilities are not discounted, the condition is: "$p(t)x(t)$ is bounded".

state that the quantities chosen along the optimal program are solutions to some appropriate individual myopic optimization problem, such as, intertemporal profit maximization or utility maximization, treating the associated prices $p(t)$ as parametric competitive market prices (see Gale and Sutherland, 1968), and, therefore, may be regarded as being decentralizable, just as in the static case. However, the requirement in (b), usually referred to as a "transversality condition", has no counterpart in static or finite horizon problems. It is observed that, being asymptotic in nature, it is not a myopic behavior rule for any individual decision maker and, therefore, it is unclear how this condition can be verified in a decentralized setting (see Koopmans, 1957; Kurz and Starrett, 1970; Hurwicz and Majumdar, 1988; Majumdar, 1988).

In a number of recent papers, several authors provide a fresh perspective on decentralization in intertemporal economies (see Brock and Majumdar, 1988; Dasgupta and Mitra, 1988; Hurwicz and Weinberger, 1990; Bala et al., 1991).[4] In several instances, alternative characterizations of optimal programs are established, whose distinguishing feature is that an asymptotic condition, such as (b), is replaced by a finite horizon condition, which can be verified at each date by myopic agents; and, therefore, optimal allocations *are* decentralizable.[5] More precisely, under fairly weak assumptions, (b) can be replaced by a variant of a rule like that in Brock and Majumdar (1988), namely, (c) $(p(t) - p^*)(x(t) - x^*) \leq 0$ for each t, where p^* and x^* denote, respectively, the prices and quantities along a stationary optimal program.

Some remarks on the results sketched above, are now in order. First, in comparison to characterizations involving an asymptotic condition like (b), as well as in comparison to the competitive conditions, the new characterizations, involving a condition like (c), typically require additional information about some other optimal program, usually a stationary program and its supporting prices. Second, while the latter involve only decentralized behaviour rules, separating consumer and producer decisions, they are not entirely satisfactory from the standpoint of incentive compatibility because, unlike the competitive conditions, a rule like (c) above does not seem to correspond to any kind of individual optimizing behaviour. Third, even from the standpoint of decentralization alone, it may be observed that, while profit maximization in the aggregate is equivalent to profit maximization at the level of individual firms, it is quite unclear how an aggregative rule like (c) can be stated in an equivalent form which is possible to verify at the level of individual firms.[6]

The remarks above highlight the fact that, while advances have been made in our understanding of decentralization in infinite horizon intertemporal economies,

[4] These, as well as several other papers on this topic, may be found in Majumdar (1992).

[5] It may be noted in passing that quite aside from the point of view of decentralization, these characterizations are also of interest from the point of view of a central planner's problem of identifying optimal and non-optimal programs. Since these new characterizations involve only period by period conditions, a non-optimal program would always be revealed to be so, within some finite horizon, by its failure to satisfy such a condition at some date. This, however, is *not* true of the characterizations involving a transversality condition since it is an asymptotic condition.

[6] These points have been noted in the literature; see Majumdar (1988), for instance.

the difference between finite and infinite horizon problems, typified by the appearance of a condition, like (b) or (c), over and above the competitive conditions, continues to be a matter of considerable interest. As Malinvaud (1992) recently observed, while it may not be justified to claim that the virtues of the price system stand or fall depending on whether one looks at a finite or infinite horizon, the validity of some central propositions in economics remains exposed to doubt and the subject merits further investigation. Broadly speaking, it is a question of interest as to what extent it is possible to weaken, or dispense with, conditions such as those in (b) or (c) above, which (in addition to the competitive conditions) are sufficient to ensure that a program is optimal.

The contribution of this paper lies in pointing out that for a fairly wide and interesting class of models the competitive conditions *alone* are sufficient to ensure optimality; that is, any additional requirement is "superfluous", whether it be in the form of an asymptotic condition, such as a transversality condition, like (b), or in an equivalent form, such as a myopic period by period condition, like (c). More precisely, in general multi-sector models, in which the technology satisfies a "reachability" property, infinite horizon programs, which satisfy the competitive conditions, are optimal (Theorem 3.1).

We should remark on the sense in which conditions, other than the competitive conditions, are superfluous. As mentioned earlier, a condition such as (b), or (c), is *both necessary and sufficient* for optimality of competitive programs; however, what turns out to be true is that, in reachable technologies, it is not independent of the competitive conditions, but rather is implied by them. Programs which are competitive *necessarily satisfy the transversality condition* (or its equivalent) and therefore are optimal.[7]

The reachability property is crucial to establishing this result, and a few remarks regarding it may be helpful at this point. McFadden (1967) was the first to formulate a notion of reachability. The notion of reachability used in this paper is in the spirit of McFadden's, but it is different.[8] The reader should consult the Reachability Condition (R) in Section 3 below for a precise statement, but the essence of Condition (R) may be paraphrased as follows: the technological production possibilities are such that, beginning with a capital stock from which expansion of stocks are feasible, it is possible (if need be through pure accumulation of capital over a sufficiently long period) to attain the stocks along any feasible program, at some future date.

In Section 4, we provide three examples (Examples 4.1 to 4.3), where the Reachability Condition (R) is satisfied and our main result applies. The examples show that the Reachability Condition is satisfied in a wide variety of multi-sector

[7] If the utility function is strictly concave then this implies that from each initial stock there is a unique competitive program which is the optimal program.

[8] McFadden is mainly concerned with closed linear models in which there are no limiting primary resources whereas we are concerned here with a model where there are limiting primary factors. His reachability property would not hold in our framework. If one disregards the difference that his condition is one on the consumption profiles along a program while ours is couched as a condition on the technology and capital stocks along programs, the reachability condition employed here is essentially a weaker version of McFadden's.

models. An example is also provided (Example 4.4) where the Reachability Condition is not satisfied and there are competitive programs which are not optimal. The example shows that something like Condition (R) is essential for Theorem 3.1 to be valid.

An informal way of summarizing our main result is that competition works in achieving optimal allocation of resources provided the technological characteristics are right. This, perhaps, may be seen as somewhat of a departure from a conventional view which holds that institutions and the legal structure of property rights are the things which enable competition to work. However, there is an extensive tradition in growth models of emphasizing technological conditions under which competition works. Solow (1956), uses capital labour substitution possibilities to show stability of competitive growth programs; Uzawa (1961–62) and Inada (1963) use capital-intensity conditions to show existence, uniqueness and stability of competitive programs in two-sector models. Malinvaud's Tightness conditions (see Malinvaud, 1953, 1962), and the various technological conditions of McFadden (1967) and Kurz (1969), which are used to show that efficiency prices are well behaved, are also prominent instances. Kurz and Starrett (1970) are interested in formulating technological properties of programs which, if they are satisfied, imply efficiency[9] of competitive programs. Our paper is very much in this tradition of relating properties of competitive programs to technology. While the evaluation criterion that Kurz and Starrett consider, as well as the specific technological conditions they formulate, are different, the main theme of their paper is similar to ours and the relationship between the two papers is of interest and is discussed in detail in Section 5. It suffices to mention here only that their *specific results* do not apply to our problem, as shown by Example 5.1 in that section.

Two final remarks are in order. First, while the methods employed in Theorem 3.1 can, with suitable modifications, deal with the case where utilities are not discounted, we deal only with the case where the utilities are discounted. Second, we are only interested in the *sufficiency* side of the price characterization of optimal programs. As is well known, the *necessity* side, (the existence of supporting prices for optimal programs) requires additional structure, in particular convexity. It is possible to verify, along the lines of Dasgupta and Mitra (1990), that standard treatments, such as Weitzman (1973), would include our model as a special case and provide the appropriate "converse" of Theorem 3.1.

2 Preliminaries

2.1 The model

The model is described by a triplet (Ω, w, δ), where Ω, a subset of $R_+^n \times R_+^n$, is the *technology set*, $w : R_+^n \to R$ is the period *welfare function*, and δ is the *discount factor*. Points in Ω are written as an ordered pair (x, y), where x stands for the

[9] For a precise definition of efficiency, see Section 5.

initial stock of inputs and y stands for the final output which can be produced with inputs x. We shall make the following assumptions[10] on (Ω, w, δ).

(A.1) There exists a number $\beta_0 > 0$ such that, if $(x, y) \in \Omega$ and $\|x\| \geq \beta_0$, then $\|y\| \leq \|x\|$.
(A.2) If $(x, y) \in \Omega$, $x' \geq x$ and $y \geq y' \geq 0$, then $(x', y') \in \Omega$.
(A.3) There is $(\hat{x}, \hat{y}) \in \Omega$ satisfying $\hat{y} \gg \hat{x}$.
(A.4) $w : R^n_+ \to R$ is continuous.
(A.5) $0 < \delta < 1$.

Note that the technology set, Ω, is not assumed to be convex, and the welfare function, w, is not assumed to be concave.

2.2 Programs

A *program* from $\tilde{y}$ in R^n_+ is a sequence $\langle x(t), y(t), c(t)\rangle$ such that $y(0) = \tilde{y}$, and

$$(x(t), y(t+1)) \in \Omega, c(t) \equiv y(t) - x(t) \geq 0 \quad \text{for all } t \geq 0.$$

We need the familiar preliminary result (see Dasgupta and Mitra, 1988) that programs from $\tilde{y}$ are uniformly bounded by a number which depends only on the initial stock $\tilde{y}$ and β_0.

Lemma 2.1. *Under* (A.1) *and* (A.2)*, if* $(x, y) \in \Omega$*, then* (i) $\|x\| \leq \beta_0$ *implies* $\|y\| \leq \beta_0$ *and* (ii) $\|y\| \leq \text{Max}\{\|x\|, \beta_0\}$.

Lemma 2.2. *Under* (A.1) *and* (A.2)*, if* $\langle x(t), y(t), c(t)\rangle$ *is a program from any* $\tilde{y}$ *in* R^n_+ *then* $(\|x(t)\|, \|y(t)\|, \|c(t)\|) \leq (\text{B}, \text{B}, \text{B})$*, for all* $t \geq 0$*, where the number* B *is defined by* $\text{B} = \text{Max}\{\|\tilde{y}\|, \beta_0\}$.

In view of Lemma 2.2, (A.4) and (A.5), it is clear that for every program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$ in R^n_+, $\sum_0^\infty \delta^t w(c(t))$ is absolutely convergent. We may, therefore, define an *optimal program* from $\tilde{y}$ in R^n_+ as a program $\langle x^*(t), y^*(t), c^*(t)\rangle$ from $\tilde{y}$ such that, $\sum_0^\infty \delta^t w(c^*(t)) \geq \sum_0^\infty \delta^t w(c(t))$ for every program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$.

A *competitive program* from $\tilde{y}$ in R^n_+ is a sequence $\langle x(t), y(t), c(t), p(t)\rangle$ such that $\langle x(t), y(t), c(t)\rangle$ is a program from $\tilde{y}$, $p(t)$ is in R^n_+ for $t \geq 0$, and the two inequalities below are satisfied:

$$\delta^t w(c(t)) - p(t)c(t) \geq \delta^t w(c) - p(t)c \text{ for all } c \geq 0, t \geq 0 \tag{2.1}$$

$$p(t+1)y(t+1) - p(t)x(t) \geq p(t+1)y - p(t)x \text{ for all } (x, y) \in \Omega, t \geq 0 \tag{2.2}$$

A competitive program is said to satisfy the *transversality condition* if $\lim_{t\to\infty} p(t)\, x(t) = 0$. It is well known that a competitive program which satisfies

[10] For x, y in R^n, $x \geq y$ means $x_i \geq y_i$ for $i = 1, \ldots, n$; $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, \ldots, n$. For x in R^n, the sum norm of x (denoted by $\|x\|$) is defined by $\|x\| = \sum_{i=1}^n |x_i|$.

the transversality condition is optimal. We state this, in Theorem 2.1 below, for ready reference.

Theorem 2.1. *Under* (A.1), (A.2), (A.4) *and* (A.5), *if* $\langle x(t), y(t), c(t), p(t)\rangle$ *is a competitive program from any* $\tilde{y}$ *in* R_+^n *and* $\lim_{t\to\infty} p(t)x(t) = 0$ *then* $\langle x(t), y(t), c(t)\rangle$ *is an optimal program from* $\tilde{y}$.

3 Optimality of competitive programs under reachability

In this section we prove the main theorem of this paper. We first introduce the reachability condition below. From now on fix a particular (x, y) in Ω with $y \gg x$ (whose existence is assumed in (A.3)) and denote it by $(\hat{x}, \hat{y})$.

Reachability Condition (R)

Given any $\tilde{y}$ in R_+^n and a program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$, there is an integer $R \geq 0$ and a program $\langle x'(t), y'(t), c'(t)\rangle$ from $\hat{y}$ such that $y'_R \geq y_R$.

We first need a basic property of competitive programs.

Lemma 3.1. *Let* $\langle x(t), y(t), c(t), p(t)\rangle$ *be a competitive program from any* $\tilde{y}$ *in* R_+^n. *Let* (x, y, c) *be any triple satisfying:* $(x, y) \in \Omega$ *and* $c \geq 0$. *Then, for* $t \geq 0$,

$$\delta^t w(c(t)) + p(t+1)y(t+1) - p(t)y(t) \geq \delta^t w(c) + p(t+1)y - p(t)(x+c) \quad (3.1)$$

Proof. Since $c \geq 0$, using (2.1), we have for $t \geq 0$

$$\delta^t w(c(t)) - p(t)c(t) \geq \delta^t w(c) - p(t)c \quad (3.2)$$

Since $(x, y) \in \Omega$, using (2.2), we have for $t \geq 0$

$$p(t+1)y(t+1) - p(t)x(t) \geq p(t+1)y - p(t)x \quad (3.3)$$

Adding (3.2) and (3.3), we have for $t \geq 0$

$$\begin{aligned} \delta^t w(c(t)) &- p(t)c(t) + p(t+1)y(t+1) - p(t)x(t) \geq \delta^t w(c) \\ &- p(t)c + p(t+1)y - p(t)x \end{aligned} \quad (3.4)$$

Since $c(t) + x(t) = y(t)$ for $t \geq 0$, we get (3.1) from (3.4). □

Theorem 3.1. *Under* (A.1) *to* (A.5) *and condition (R), if* $\langle x(t), y(t), c(t), p(t)\rangle$ *is a competitive program from any* $\tilde{y}$ *in* R_+^n, *then* $\langle x(t), y(t), c(t)\rangle$ *is an optimal program from* $\tilde{y}$.

Proof. Define $\hat{c} = 0$ and apply (3.1) to the triple $(\hat{x}, \hat{y}, \hat{c})$, to get,

$$\begin{aligned} &\delta^t w(c(t)) + p(t+1)y(t+1) - p(t)y(t) \geq \delta^t w(\hat{c}) + p(t+1)\hat{y} - p(t)(\hat{x} + \hat{c}) \\ &= \delta^t w(0) + p(t+1)\hat{y} - p(t)\hat{x} = \delta^t w(0) + p(t+1)(\hat{y} - \hat{x}) + p(t+1)\hat{x} - p(t)\hat{x} \end{aligned} \quad (3.5)$$

Then for any $T \geq 2$,

$$\begin{aligned}\sum_0^{T-1} \delta^t w(c(t)) + \quad & p(T)y(T) - p(0)y(0) \\ \geq \quad & \sum_0^{T-1} \delta^t w(0) + \sum_0^{T-1} p(t+1)(\hat{y} - \hat{x}) + p(T)\hat{x} - p(0)\hat{x} \quad (3.6) \\ = \quad & \sum_0^{T-1} \delta^t w(0) + \sum_0^{T-2} p(t+1)(\hat{y} - \hat{x}) + p(T)\hat{y} - p(0)\hat{x}\end{aligned}$$

Consider the sequence $\langle x''(t), y''(t), c''(t)\rangle$ defined by $(x''(t), y''(t), c''(t)) = (x(t+T), y(t+T), c(t+T))$ for $t \geq 0$. Then, clearly, $\langle x''(t), y''(t), c''(t)\rangle$ is a program from $y(T)$. By condition (R), there is a program $\langle x'(t), y'(t), c'(t)\rangle$ from $\hat{y}$ and $R \geq 0$ such that

$$y'(R) \geq y''(R) = y(T+R) \quad (3.7)$$

Applying (3.1) to $(x'(s), y'(s+1), c'(s))$ we have, for $s \geq 0$,

$$\begin{aligned}&\delta^{T+s} w(c(T+s)) + p(T+s+1)y(T+s+1) - p(T+s)y(T+s) \\ &\quad \geq \delta^{T+s} w(c'(s)) + p(T+s+1)y'(s+1) - p(T+s)(x'(s) + c'(s)) \\ &\quad = \delta^{T+s} w(c'(s)) + p(T+s+1)y'(s+1) - p(T+s)y'(s) \quad (3.8)\end{aligned}$$

In what follows it is understood that if $R = 0$ then whenever a sum from 0 to $R - 1$ appears it is, by convention, taken to be equal to 0.

Summing from $s = 0$ to $s = R - 1$, (3.8) yields

$$\begin{aligned}\sum_0^{R-1} \delta^{T+s} w(c(T+s)) + \; & p(T+R)y(T+R) \\ - \; p(T)y(T) \geq & \sum_0^{R-1} \delta^{T+s} w(c'(s)) + p(T+R)y'(R) - p(T)y'(0)\end{aligned} \quad (3.9)$$

Using (3.7), $p(t) \geq 0$ for all $t \geq 0$, and $y'(0) = \hat{y}$ in (3.9) we get

$$\sum_0^{R-1} \delta^{T+s} w(c(T+s)) - p(T)y(T) \geq \sum_0^{R-1} \delta^{T+s} w(c'(s)) - p(T)\hat{y} \quad (3.10)$$

Using (3.6) and (3.10), for $T \geq 2$, we get

$$\begin{aligned}\sum_0^{T+R-1} \delta^t w(c(t)) \quad - \quad & p(0)y(0) \\ \geq \quad & \sum_0^{T-1} \delta^t w(0) + \sum_0^{R-1} \delta^{T+s} w(c'(s)) \quad (3.11) \\ + \quad & \sum_0^{T-2} p(t+1)(\hat{y} - \hat{x}) - p(0)\hat{x}\end{aligned}$$

Since $\langle x(t), y(t), c(t)\rangle$ is a program from $\tilde{y}$, by Lemma 2.2, $\|c(t)\| \leq B$, where $B = \max\{\beta_0, \|\tilde{y}\|\}$. Since $\langle x'(t), y'(t), c'(t)\rangle$ is a program from $\hat{y}$, $\|c'(t)\| \leq B_1$, where $B_1 = \max\{\beta_0, \|\hat{y}\|\}$. Define $B^* = \max\{B_1, B\}$, and $B_2 = \underset{0\leq c\leq B^*e}{\text{Max}} |w(c)|$ where $e = [1, 1, \ldots, 1]$ is in R^n.

Using this notation, from (3.11), we get

$$\begin{aligned}\sum_0^{T-2} p(t+1)(\hat{y} - \hat{x}) \quad &\leq \quad \sum_0^{T+R-1} \delta^t w(c(t)) - \sum_0^{T-1} \delta^t w(0) \\ &\quad - \sum_0^{R-1} \delta^{T+s} w(c'(s)) + p(0)(\hat{x} - y(0)) \\ &\leq \quad 3B_2(1-\delta)^{-1} + p(0)(\hat{x} - y(0)) < \infty\end{aligned}$$

We have established that, for any $T \geq 2$, $\sum_0^{T-2} p(t+1)(\hat{y} - \hat{x}) < \infty$. Since $\hat{y} - \hat{x} \gg 0$, and $p(t) \geq 0$, we can infer that $\sum_0^{\infty} p(t) < \infty$ and, consequently, $p(t) \to 0$ as $t \to \infty$. Since $x(t)$ is bounded (by Lemma 2.2), $p(t)x(t) \to 0$ as $t \to \infty$. Thus, by Theorem 2.1, $\langle x(t), y(t), c(t)\rangle$ is optimal. □

4 Examples

4.1 Summary descriptions

In this section we provide four examples. The first three (in Subsection 4.1 below) are examples where the assumptions on the technology (A.1)–(A.3), and the Reachability Condition (R) is satisfied and Theorem 3.1 applies (whenever the assumptions on the welfare function (A.4) and the discount factor (A.5) are met). The examples show that Condition (R) can be satisfied in a wide variety of production models. In all of the examples there is only one exogenously given (non-produced or primary) factor of production (labor). However, as will be evident from the detailed descriptions of the examples to follow, this is not an essential feature and, with obvious modifications, more than one primary factor may be allowed.

In Example 4.1, the production side is the Simple Leontief Model (see Gale, 1960) where, corresponding to each good, there is one fixed coefficient production process. It does not allow for choice of technique, nor for the possibility of joint production. Example 4.2 does allow for choice of technique and joint production. It is a General Linear Model with a finite number of fixed coefficient production processes (see Gale, 1960). Example 4.3 is a non-linear production model allowing for variable coefficients. It is a two-sector model where the technology in each sector is described by a smooth neoclassical production function.

In Example 4.1, provided the technology satisfies the productivity assumption (A.3) of Section 2, the Reachability Condition (R) is satisfied. In Examples 4.2 and 4.3, stronger productivity assumptions are made to ensure Reachability. No attempt is made to characterize Condition (R) in terms of technological coefficients.

That some condition like (R) is needed, in a general model, in addition to the assumptions (A.1) to (A.5), to obtain the conclusion of Theorem 3.1, is demonstrated by the final example (Example 4.4 in Subsection 4.3 below). The example is a two-sector linear model with two processes which allow joint production. Here, the assumptions (A.1) to (A.5) of Section 2 are satisfied but Condition (R) is not satisfied. We construct a program which is shown to be competitive but not optimal.

4.2 Examples of models where the reachability condition is satisfied

Example 4.1: Simple leontief model

In this example, Ω is generated by a square matrix A of order n and a vector a in R^n (see (L.1) below). As before, $w : R^n_+ \to R$ denotes the welfare function and δ the discount factor. The following assumptions are made:

(L.1) There is an $n \times n$ real matrix $A = [a_{ij}], i = 1, \ldots, n, j = 1, \ldots, n$ and a vector $a = (a_1, \ldots, a_n)$ in R^n such that for any (x, y) in $R^n \times R^n, (x, y) \in \Omega$ iff

$$x \geq Ay, x \geq 0, y \geq 0, \text{ and } ay \leq 1. \tag{4.1}$$

(L.2) $A \geq 0$, that is, $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$; $a \gg 0$.

(L.3) A is productive; that is, there is $\hat{y} \gg 0$ such that $\hat{y} \gg A\hat{y}$ and $a\hat{y} \leq 1$. Here, a_{ij} and a_j are, respectively, the amounts of the i^{th} good and labor which are required to produce one unit of output of the j^{th} good.

Lemma 4.1. (a) *If Ω satisfies* (L.1) *to* (L.3) *then Ω satisfies* (A.1) *to* (A.3) *of Section 2 and, in particular, if $(x, y) \in \Omega$ then $y \leq \beta_0 e$, where $\beta_0 = 1/\min_i a_i$ and $e = [1, \ldots, 1]$*; (b) *If Ω satisfies* (L.1) *and* (L.2), *then Ω satisfies* (A.3) *iff it satisfies* (L.3).

Proof. (a) Suppose that Ω satisfies (L.1) to (L.3). Then clearly Ω satisfies (A.2). Since $a \gg 0$, we may define a number β_0 by $\beta_0 = 1/\operatorname{Min}_i\{a_i\}$. Then, it follows from (4.1) that for all $(x, y) \in \Omega$, we must have $y \leq \beta_0 e$. This establishes that Ω satisfies (A.1). Finally, define $\hat{x}$ by $\hat{x} = A\hat{y}$. Then it is clear that $(\hat{x}, \hat{y}) \in \Omega$ and, using (L.3), $\hat{x} \ll \hat{y}$. This shows that (A.3) is also satisfied. (b) Suppose that Ω satisfies (L.1) and (L.2). Then it is obvious from the definitions that if Ω satisfies (A.3) then it also satisfies (L.3). Together with part (a), this establishes part (b). □

Remark 4.1. It is well known (see Gale, 1960) that if A is productive and $A \geq 0$ then $A^t \to 0$ as $t \to \infty$, and $(I - A)^{-1} = \sum_0^\infty A^t$, where, by convention, A^0 is the identity matrix I.

We now show that under (L.1)–(L.3), the Reachability Condition (R) is satisfied.

Lemma 4.2. *Under* (L.1) *to* (L.3), *if $\langle x(t), y(t), c(t) \rangle$ is a program from any $\hat{y}$ in R^n_+ then there is a program $\langle x'(t), y'(t), c'(t) \rangle$ from $\hat{y}$ and an integer $R \geq 0$ such that $y'(R) = y(R)$.*

Proof. Suppose that $\langle x(t), y(t), c(t)\rangle$ is a program from $\tilde{y}$ in R^n_+. From Lemma 4.1, we have $y(t) \le \beta_0 e$ for all $t \ge 1$. From $\hat{y} \gg 0$ and $A^t \to 0$ (see Remark 4.1) we have an integer $R \ge 2$ such that

$$A^R \beta_0 e \ll \hat{y} \tag{4.2}$$

Now define the sequence $\langle x'(t), y'(t), c'(t)\rangle$ as follows: $y'(0) = x'(0) = \hat{y}$,

$$x'(t) = A^{R-t} y(R) = y'(t) \text{ for } 1 \le t \le R-1 \tag{4.3}$$

$x'(t) = x(t)$ for all $t \ge R$, $y'(t) = y(t)$ for all $t \ge R$, $c'(t) = y'(t) - x'(t)$ for all $t \ge 0$.

Clearly $x'(t) \ge 0$, and $y'(t) \ge 0$ for all $t \ge 0$, $c'(t) = 0$ for $0 \le t \le R-1$ and $c'(t) = c(t) \ge 0$ for $t \ge R$. Also, by definition, $y'(0) = \hat{y}$; so, to verify that $\langle x'(t), y'(t), c'(t)\rangle$ is a program from $\hat{y}$, we need only to verify that

$$Ay'(t+1) \le x'(t) \text{ for all } t \ge 0 \tag{4.4}$$

and that

$$ay'(t) \le 1 \text{ for all } t \ge 1 \tag{4.5}$$

Clearly (4.4) and (4.5) hold for all $t \ge R$, because, for such t, the sequence $\langle x'(t), y'(t), c'(t)\rangle$ coincides with the given sequence $\langle x(t), y(t), c(t)\rangle$ which satisfies these inequalities, being a program.

Now, since $A \ge 0$, we get $y'(1) = A^{R-1} y(R) \le A^{R-1}\beta_0 e$. So, using (4.2), $Ay'(1) \le A^R \beta_0 e \ll \hat{y} = x'_0$. This establishes (4.4) for $t = 0$. If $1 \le t < R-1$, then, using (4.3), we get $Ay'(t+1) = AA^{R-(t+1)} y(R) = x'(t)$. This verifies (4.4) for t satisfying $1 \le t < R-1$. Finally, for $t = R-1$, $Ay'(t+1) = Ay'(R) = Ay(R) = A^{R-(R-1)} y(R) = x'(R-1) = x'(t)$. This verifies that (4.4) holds for all $t \ge 0$.

We will now verify that (4.5) holds along $\langle x'(t), y'(t), c'(t)\rangle$ for $1 \le t \le R-1$. Since $\langle x(t), y(t), c(t)\rangle$ is a program, $Ay(t+1) \le x(t) \le y(t)$ for $t \ge 0$. Thus, for any integer $\theta \ge 1$, $A^\theta y(t+\theta) = A^{\theta-1}(Ay(t+\theta)) \le A^{\theta-1} y(t+\theta-1)$ for all $t \ge 0$. Applying this repeatedly, and noting that $A^\circ = I$, we have

$$A^\theta y(t+\theta) \le y(t) \text{ for all } t \ge 0, \text{ and any integer } \theta \ge 1 \tag{4.6}$$

So, for $1 \le t \le R-1$ we have, from (4.3), after setting $\theta = R-t$ in (4.6), $y'(t) = A^{R-t} y(R) \le y(t)$. Since $a \ge 0$ and $ay(t) \le 1$ for all $t \ge 1$, we have $ay'(t) \le ay(t) \le 1$ for $1 \le t \le R-1$. This verifies (4.5) and completes the proof of the lemma. □

Example 4.2: General linear model

In this example the technology set Ω is defined by two $n \times m$ matrices A and B and a vector a in R^m. The following assumptions are made:

(GL.1) There are two $n \times m$ matrices $A = [a_{ij}]$, and $B = [b_{ij}]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and a vector $a = (a_1, \ldots, a_m)$ in R^m such that (x, y) is in Ω iff there is z in R^m satisfying

$$x \geq Az \geq 0,\ Bz \geq y \geq 0,\ z \geq 0,\ az \leq 1 \tag{4.7}$$

(GL.2) $A \geq 0$, $B \geq 0$; $a \gg 0$

Here z_j denotes the level of the j^{th} activity or process, a_{ij} is the amount of the i^{th} good required, a_j is the amount of labor required, and b_{ij} is the output of the i^{th} good produced, per unit of activity j. It is clear that, when Ω satisfies (GL.1), corresponding to any program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$ in R^n_+, we can associate a sequence $\langle z(t)\rangle$ such that, for each $t \geq 0, (x, y, z) = (x(t), y(t+1), z(t+1))$ satisfies (4.7). We shall say that $z(t)$ is the associated sequence of activity levels along the given program. In the remainder of this section it would be convenient to choose the normalization for the processes in which $a = e = [1, \ldots, 1]$.[11]

We need to make a strong form of the productivity assumption ((GL.3) below). Let $S = \{z \text{ in } R^m | z \geq 0 \text{ and } \sum_{j=1}^m z_j = az = 1\}$ be the unit simplex in R^m.

(GL.3) (*Strong Productivity*) For each $j = 1, \ldots, m$ there is π^j in S satisfying $B\pi^j \gg a^j$.

Assumption (GL.3) says that corresponding to the input stock (vector) of any basic process there is a strictly larger feasible output, producible from *some* input stock.

It is posssible to show that under assumptions (GL.1) to (GL.3), the technology set Ω satisfies (A.1) to (A.3) as well as the Reachability Condition (R).[12]

Example 4.3: A non-linear model

Let X_{ij} denote the amount of the i^{th} produced good used as input in industry j, where $i \neq j$; let L_j denote the amount of labor used, and Y_j the output produced, in industry j. The production functions of the two industries are denoted by $F^1(X_{21}, L_1)$ and $F^2(X_{12}, L_2)$. In this example the technology set Ω is defined by

$$\begin{aligned}\Omega = \{&(X_1, X_2, Y_1, Y_2) \in R^4_+ | \text{ there is a } (X_{12}, X_{21}, L_1, L_2, Y_1, Y_2) \in R^6_+ \\ &\text{satisfying } Y_1 \leq F^1(X_{21}, L_1), Y_2 \leq F^2(X_{12}, L_2), X_{12} \leq X_1, X_{21} \leq X_2, \\ &\text{and } L_1 + L_2 \leq 1\}\end{aligned}$$

Assume that:

(NL.1) For each $j = 1, 2$, F^j is twice continuously differentiable, concave and homogeneous of degree 1.

(NL.2) For each $j = 1, 2$, F^j is non-negative and has non-negative partial derivatives; furthermore, $F^j(X_{ij}, 0) = 0 = F^j(0, L_j)$; $F^j(X_{ij}, L_j) > 0$ when $X_{ij} > 0$, $L_j > 0$.

Define the normalized production functions f^j by $f^j(X_{ij}) = F^j(X_{ij}, 1)$ for $j = 1, 2$. Assume that, in addition to (NL.1) and (NL.2), Ω satisfies the following productivity assumption:

[11] The unit levels of the processes are being defined so that labor required per unit is 1. This is simply a matter of convenience, and not a restriction beyond assuming that labor required in each process is strictly positive (see (GL.2)).

[12] The proofs are lengthy, and consequently omitted; they are available upon request from the authors.

(NL.3) For each $j = 1, 2$, there is a point $\overline{x}_j$ at which $f^j(.)$ reaches its maximum value, denoted by $\overline{y}_j$; moreover $(\overline{y}_1/\overline{x}_2) > (\overline{x}_1/\overline{y}_2)$.[13]

It is possible to show that under (NL.1)–(NL.3), assumptions (A.1) to (A.3), as well as the Reachability Condition (R) are satisfied.[14]

4.3 Example where the reachability condition is not satisfied

We now provide an example[15] of a two good model with fixed coefficient production processes, allowing joint production but with the number of activities equal to the number of goods, where the Reachability Condition (R) fails to hold and there are competitive programs which are not optimal. The example illustrates that the Reachability Condition fairly precisely delineates the class of technologies, within the class of General Linear Models, for which the transversality condition can be dispensed with.

Example 4.4: Choose any γ satisfying $0 < \gamma < 1$. Define $w : R^2_+ \to R$ by

$$w(c_1, c_2) = c_1^{\gamma} + c_2 \text{ for } c = (c_1, c_2) \geq 0$$

Define Ω as in (GL.1) of Example 4.2 where the matrices A, B, and a are chosen as follows:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},\ a = (1, 1)$$

Define $\hat{z} = [0.5, 0.5]$. Then $A\hat{z} = [2, 2]$ and $B\hat{z} = [2.5, 2.5]$. Define $(\hat{x}, \hat{y}) = (A\hat{z}, B\hat{z})$; then $(\hat{x}, \hat{y})$ satisfies (A.3). Take any δ satisfying $0 < \delta < 1$. Then, it is clear that (A.1)–(A.5) hold.

Now, define a sequence $\langle x(t), y(t), c(t), z(t)\rangle$ by $z(t) = [e(t), 1 - e(t)]$, $y(t) = Bz(t)$, $x(t) = Az(t + 1)$, $c(t) = [1, 0]$ for $t \geq 0$, where we use $e(t)$ to denote $(1/2^t)$, to simplify notation.

We shall first verify that $\langle x(t), y(t), c(t)\rangle$ is a program from [3, 2]. It is straightforward to check that $y(0) = [3, 2]$, and $(x(t), y(t + 1))$ is in Ω for $t \geq 0$. It remains to check that $y(t) - x(t) = c(t)$ for $t \geq 0$. Note that $y(t) = [3e(t) + 2(1 - e(t)),\ 2e(t) + 3(1 - e(t))] = [2 + e(t), 3 - e(t)]$ for $t \geq 0$. Also, $x(t) = [3e(t+1)+(1-e(t+1)), e(t+1)+3(1-e(t+1))] = [1+2e(t+1), 3-2e(t+1)] = [1 + e(t), 3 - e(t)]$ for $t \geq 0$. Thus, $y(t) - x(t) = [1, 0] = c(t)$ for $t \geq 0$.

Next, we define prices $(p(t), w(t)) = (p_1(t), p_2(t), w(t))$ supporting the program. Let $p_2(0) = \text{Max}\,\{3\gamma, 1\}$, and define for $t \geq 0$:

[13] (NL.3) says that the production functions, with labor requirement normalized to equal unity, are bounded above and attain their upper bounds; furthermore, the Simple Leontief Model composed of the two normalized processes, where the respective upper bounds are attained, satisfies the productivity condition (see (L.3) in Example 4.1) expressed in the form of the Hawkins-Simon condition (see Hawkins and Simon, 1949).

[14] Details are available upon request from the authors.

[15] See Dasgupta and Mitra (1993) for additional examples of interest.

$$p_1(t) = \delta^t\gamma, p_2(t+1) = p_1(t+1) - 2p_1(t) + 2p_2(t) \quad (4.8)$$
$$w_t = 5p_1(t+1) - 7p_1(t) + 3p_2(t) \quad (4.9)$$

We verify that $(p(t), w(t)) \geq 0$ for $t \geq 0$. Notice that $p_1(t) > 0$ for $t \geq 0$ by (4.8). Also, by (4.8), we have

$$(p_2(t+1) - p_1(t+1)) = 2(p_2(t) - p_1(t)) \text{ for } t \geq 0 \quad (4.10)$$

Since $(p_2(0) - p_1(0)) \geq 3\gamma - \gamma = 2\gamma > 0$, we obtain

$$(p_2(t) - p_1(t)) \geq 2^t(2\gamma) \text{ for } t \geq 0 \quad (4.11)$$

Thus $p_2(t) > 0$ for $t \geq 0$. Also, for $t \geq 0$, $w(t) = 5p_1(t+1) - 7p_1(t) + 3p_2(t) > 3(p_2(t) - p_1(t)) - 4p_1(t) \geq 6\gamma - 4\gamma$ (by using (4.11) $= 2\gamma > 0$.

We check now that $\langle x(t), y(t), c(t), p(t)\rangle$ satisfies the profit-maximizing condition (2.2). For $t \geq 0$, we have $p(t+1)B - p(t)A - w(t)a = (3p_1(t+1) + 2p_2(t+1) - 3p_1(t) - p_2(t) - w(t), 2p_1(t+1) + 3p_2(t+1) - p_1(t) - 3p_2(t) - w(t)) = (3p_1(t+1) + 2(p_1(t+1) - 2p_1(t) + 2p_2(t)) - 3p_1(t) - p_2(t) - w(t), 2p_1(t+1) + 3(p_1(t+1) - 2p_1(t) + 2p_2(t)) - p_1(t) - 3p_2(t) - w(t)) = (5p_1(t+1) - 7p_1(t) + 3p_2(t) - w(t), 5p_1(t+1) - 7p_1(t) + 3p_2(t) - w(t)) = (0, 0)$ for all $t \geq 0$, using (4.8) and (4.9). It is now straightforward to verify, using (4.7), that for any (x, y) in Ω and $t \geq 0$, $p(t+1)y - p(t)x \leq w(t) = p(t+1)y(t+1) - p(t)x(t)$. This establishes (2.2).

Finally, we check that $\langle x(t), y(t), c(t), p(t)\rangle$ satisfies the condition (2.1). To this end, note that by using (4.11), $(p_2(t) - p_1(t)) \geq 4\gamma$ for all $t \geq 1$. This yields $p_2(t) \geq 3\gamma$ for all $t \geq 0$, and so $(p_2(t) - 2p_1(t)) \geq (p_2(t) - 2p_1(0)) \geq (3\gamma - 2\gamma) = \gamma > 0$. Using this information in (4.8) yields $p_2(t+1) = p_1(t+1) + (p_2(t) - 2p_1(t)) + p_2(t) > p_2(t)$ for $t \geq 0$, and since $p_2(0) \geq 1$, we infer that $p_2(t) \geq 1$ for $t \geq 0$.

Now, for any $t \geq 0$, and any c in R^2_+, $\delta^t w(c) - p(t)c = [\delta^t(c_1)^\gamma - p_1(t)c_1] + [\delta^t c_2 - p_2(t)c_2] \leq [\delta^t(c_1)^\gamma - p_1(t)c_1]$ (using $p_2(t) \geq 1 \geq \delta^t$ for $t \geq 0$) $= \delta^t[(c_1)^\gamma - (c_1(t))^\gamma] + [\delta^t(c_1(t))^\gamma - p_1(t)c_1(t)] + p_1(t)[c_1(t) - c_1] \leq p_1(t)[c_1 - c_1(t)] + [\delta^t(c_1(t))^\gamma - p_1(t)c_1(t)] + p_1(t)[c_1(t) - c_1]$ (using concavity of the function c^γ on R_+, $c_1(t) = 1$, and $p_1(t) = \delta^t\gamma$ for $t \geq 0$) $= [\delta^t(c_1(t))^\gamma - p_1(t)c_1(t)] = \delta^t w(c(t)) - p(t)c(t)$ (since $c_2(t) = 0$ for $t \geq 0$). This verifies (2.1), and establishes that $\langle x(t), y(t), c(t), p(t)\rangle$ is a competitive program.

Finally, we verify that the program $\langle x(t), y(t), c(t)\rangle$ is *not* optimal. To see this, define $y'(0) = y(0)$, $c'(0) = c(0)$, $(x'(t), y'(t+1)) = (\hat{x}, \hat{y})$ for $t \geq 0$, and $c'(t+1) = [0.5, 0.5]$ for $t \geq 0$. Then, clearly, $\langle x'(t), y'(t), c'(t)\rangle$ is a program from [3, 2]. Also,

$$\sum_{t=0}^{\infty} \delta^t w(c'(t)) = 1 + \sum_{t=1}^{\infty} \delta^t[(1/2^\gamma) + (1/2)] > 1 + \sum_{t=1}^{\infty} \delta^t = \sum_{t=0}^{\infty} \delta^t = \sum_{t=0}^{\infty} \delta^t w(c(t))$$

So, $\langle x(t), y(t), c(t)\rangle$ is not optimal from [3, 2].

It follows, of course, that in this example the Reachability Condition (R) is not satisfied. This may also be verified directly.

We conclude this section by noting that the example is “borderline” in the following sense. Suppose we introduce a perturbation $\epsilon > 0$, in the output matrix B of the example, as follows: for each $j = 1, 2$, the output of the j^{th} good from the j^{th} activity is ϵ more compared to before, where ϵ is allowed to be as small as we wish. So the new B matrix is given by $\begin{bmatrix} 3+\epsilon & 2 \\ 2 & 3+\epsilon \end{bmatrix}$. Then it may be verified that the example becomes a special case of Example 4.2, because the Strong Productivity assumption (GL.3) will be satisfied. As a consequence, Condition (R) will be satisfied and all competitive programs will be optimal in the perturbed example.

5 Locally expandable and contractable programs and reachability

In this section, we relate the results of this paper to the contribution of Kurz and Starrett (1970).[16]

It will be easier to conduct such a discussion if we first define a few concepts involved in their paper. A program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$ in R^n_+ is called *efficient* if there is no other program $\langle x'(t), y'(t), c'(t)\rangle$ from $\tilde{y}$ satisfying $c'(t) \geq c(t)$ for all $t \geq 0$ and $c'(t) > c(t)$ for some $t \geq 0$. Further, define a sequence $\langle x(t), y(t), c(t), p(t)\rangle$ to be an *intertemporal profit-maximizing program* if $\langle x(t), y(t), c(t)\rangle$ is a program from $y(0)$, $p(t)$ is in R^n_+ for $t \geq 0$, and condition (2.2) is satisfied.[17]

The objective of Kurz-Starrett’s paper is to obtain technological conditions under which intertemporal profit-maximizing programs satisfy the transversality condition $\lim_{t\to\infty} p(t)x(t) = 0$, and are, therefore, (using the basic result of Malinvaud, 1953) efficient. The theme of their paper is clearly similar to ours, since we seek to obtain conditions on the technology under which competitive programs satisfy the transversality condition and are, therefore, optimal.[18]

The specific technological conditions proposed by Kurz-Starrett can be reformulated in the context of the General Linear Model[19] (that is, when Ω satisfies (GL.1) and (GL.2) of Section 4) as follows.

[16] Since writing this paper, it has come to our attention that in the literature on money and overlapping generations model there have appeared papers which address issues of dynamic inefficiencies in such models and are therefore similar in spirit to Kurz-Starrett. The problems and results, however, are different and have no direct bearing on our problem. The interested reader may consult Rhee (1991) and references cited there.

[17] Kurz-Starrett call such programs *competitive*. Since we use the term *competitive* to denote programs with price supports, which satisfy both (2.1) and (2.2), it is useful for our discussion to designate programs as intertemporal profit maximizing when they are required to satisfy only (2.2).

[18] Kurz and Starrett (1970, p. 576) observe “If a technology is sufficiently special, then all programmes may satisfy these conditions and thus all competitive programmes will be efficient”. Whether there actually exist technologies in which *all* intertemporal profit maximizing programs are efficient, is an open question. In this respect, our exercise is more complete: we give several examples of technologies in which our Reachability Condition (R) is satisfied by *all* competitive programs, which are consequently also optimal.

[19] The production side of the model in Section 2 is a special case of the very general framework of Kurz-Starrett.

Condition LE: A program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$ in R^n_+ is *Locally Expandable* if there is an $\epsilon > 0$, an integer $T \geq 0$ and a program $\langle x'(t), y'(t), c'(t)\rangle$ from some y' in R^n_+ such that

$$y'(t) = (1+\epsilon)y(t) \quad \text{for all } t > T \tag{5.1}$$

$$c'(t) = (1+\epsilon)c(t) \quad \text{for all } t > T \tag{5.2}$$

$$az'(t) = az(t) \quad \text{for all } t > T \tag{5.3}$$

where $z(t)$ and $z'(t)$, respectively, are the activity levels along the two programs.

Condition LC: A program $\langle x(t), y(t), c(t)\rangle$ from $\tilde{y}$ in R^n_+ is *Locally Contractable* if there is $\epsilon > 0$ and $\alpha > 0$ such that for each $t \geq 0$ there is $z'(t+1) \geq 0$ satisfying

$$Bz'(t+1) \geq y(t+1) \tag{5.4}$$

$$Az'(t+1) \leq (1-\alpha)x(t) \tag{5.5}$$

$$az'(t+1) \leq 1+\epsilon \tag{5.6}$$

Consider, now, a competitive program $\langle x(t), y(t), c(t), p(t)\rangle$. If the program $\langle x(t), y(t), c(t)\rangle$ is locally expandable or locally contractable, then by the results of Kurz and Starrett (1970), the transversality condition $\lim_{t\to\infty} p(t)x(t) = 0$ would be satisfied. Consequently, by Theorem 2.1, it would be optimal.

We now provide an example (essentially the Simple Leontief Model discussed in Section 4) in which our Reachability Condition (R) is satisfied and so, by Theorem 3.1, all competitive programs are optimal. But, here, the results of Kurz-Starrett cannot be applied to obtain this conclusion, because competitive programs in this framework are neither locally expandable nor locally contractable.

Example 5.1. Let the production side be the Simple Leontief Model (see Example 4.1 in Section 4); that is, assume that Ω satisfies (L.1) to (L.3). Note that this may be viewed as a special case of the General Linear Model (see the description of Ω in (GL.1) and (GL.2) of Example 4.2) with the restriction that the output matrix $B = I$ and the vector of activity levels z is identified with the output vector y. Assume also that $w : R^n_+ \to R$ satisfies:

(L.4) (a) w is continuous and concave on R^n_+ with $w(0) = 0 < w(e)$ where $e = [1, \ldots, 1]$ in R^n.
(b) $w(c') \geq (>) w(c)$ for $c' \geq (>) c$.
(c) $(w(\lambda e)/\lambda) \to \infty$ as $\lambda \to 0$.

Clearly, (L.4) ensures that (A.4) is satisfied. An example of w satisfying (L.4) is

$$w(c_1, \ldots, c_n) = \sum_{i=1}^{n} c_i^{\gamma_i} \text{ where } 1 > \gamma_i > 0 \text{ for } i = 1, \ldots, n.$$

(L.5) (a) $0 < \delta < 1$
(b) For each $j = 1, \ldots, n$, the column vector $a^j > 0$.

The condition (L.5)(b) is standard for Leontief technologies. It says that to produce (a positive amount of) any good j, ($j = 1, \ldots, n$) one needs a positive amount of *some* good i ($i = 1, \ldots, n$), apart from the labor requirement.

It may be noted at this point that the example is a special case of the framework considered in Dasgupta and Mitra (1990, Section 3). Thus, there is an optimal program from every $\tilde{y}$ in R^n_+, and there is a competitive program from every $\tilde{y}$ in R^n_{++}.

Let $\langle x(t), y(t), c(t), p(t)\rangle$ be *any* competitive program from $\tilde{y} \gg 0$. We claim that the program $\langle x(t), y(t), c(t)\rangle$ can be neither Locally Expandable nor Locally Contractable.

To establish the claim, we first note a few basic properties of the competitive program. Since, by our analysis of Section 4, the program satisfies the Reachability Condition (R), it is optimal (by Theorem 3.1). Letting $\langle z(t)\rangle$ denote the activity level sequence associated with the program, we have $z(t+1) \geq y(t+1)$, $Az(t+1) \leq x(t)$ for $t \geq 0$. Since $\langle x(t), y(t), c(t)\rangle$ is optimal, we must in fact have $z(t+1) = y(t+1)$ and $Az(t+1) = x(t)$ for $t \geq 0$, by using (L.4)(b). Since $\langle x(t), y(t), c(t), p(t)\rangle$ is competitive, condition (2.1) implies that $c(t) > 0$ for $t \geq 0$ by using (L.4)(c). Thus $z(t) = y(t) \geq c(t) > 0$ for $t \geq 1$, and $x(t) = Az(t+1) > 0$ for $t \geq 0$ by using (L.5).

If $\langle x(t), y(t), c(t)\rangle$ were *locally expandable*, then using conditions (5.1) and (5.3), we would get for $t > T$

$$ay(t) = az(t) = az'(t) \geq ay'(t) = (1+\epsilon)ay(t)$$

which implies that $ay(t) = 0$, a contradiction since $y(t) > 0$ and $a \gg 0$.

If $\langle x(t), y(t), c(t)\rangle$ were *locally contractable*, then $z'(t+1) \geq y(t+1)$ from (5.4), and using this in (5.5), $Ay(t+1) \leq Az'(t+1) \leq (1-\alpha)x(t)$. But since $Ay(t+1) = Az(t+1) = x(t)$ for $t \geq 0$, we must have $x(t) \leq (1-\alpha)x(t)$, so that $x(t) = 0$, a contradiction.

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